

# ON THE SUM OF DIGITS OF THE FACTORIAL

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**ABSTRACT.** Let  $b \geq 2$  be an integer and denote by  $s_b(m)$  the sum of the digits of the positive integer  $m$  when is written in base  $b$ . We prove that  $s_b(n!) > C_b \log n \log \log \log n$  for each integer  $n > e$ , where  $C_b$  is a positive constant depending only on  $b$ . This improves of a factor  $\log \log \log n$  a previous lower bound for  $s_b(n!)$  given by Luca. We prove also the same inequality but with  $n!$  replaced by the least common multiple of  $1, 2, \dots, n$ .

## 1. INTRODUCTION

Let  $b \geq 2$  be an integer and denote by  $s_b(m)$  the sum of the digits of the positive integer  $m$  when is written in base  $b$ . Lower bounds for  $s_b(m)$  when  $m$  runs through the member of some special sequence of natural numbers (e.g., linear recurrence sequences [Ste80] [Luc00] and sequences with combinatorial meaning [LS10] [LS11] [KL12] [Luc12]) have been studied before.

In particular, Luca [Luc02] showed that the inequality

$$(1) \quad s_b(n!) > c_b \log n,$$

holds for all the positive integers  $n$ , where  $c_b$  is a positive constant, depending only on  $b$ . He also remarked that (1) remains true if one replaces  $n!$  by

$$\Lambda_n := \text{lcm}(1, 2, \dots, n),$$

the least common multiple of  $1, 2, \dots, n$ . We recall that  $\Lambda_n$  has an important role in elementary proofs of the Chebyshev bounds  $\pi(x) \asymp x / \log x$ , for the prime counting function  $\pi(x)$  [Nai82].

In this paper, we give a slight improvement of (1) by proving the following

**Theorem 1.1.** *For each integer  $n > e$ , it results*

$$s_b(n!), s_b(\Lambda_n) > C_b \log n \log \log \log n,$$

where  $C_b$  is a positive constant, depending only on  $b$ .

## 2. PRELIMINARIES

In this section, we discuss a few preliminary results needed in our proof of Theorem 1.1. Let  $\varphi$  be the Euler's totient function. We prove an asymptotic formula for the maximum of the preimage of  $[1, x]$  through  $\varphi$ , as  $x \rightarrow +\infty$ . Although the cardinality of the set  $\varphi^{-1}([1, x])$  is well studied [Bat72] [BS90] [BT98], in the literature we have found no results about  $\max(\varphi^{-1}([1, x]))$  as our next lemma.

**Lemma 2.1.** *For each  $x \geq 1$ , let  $m = m(x)$  be the greatest positive integer such that  $\varphi(m) \leq x$ . Then  $m \sim e^\gamma x \log \log x$ , as  $x \rightarrow +\infty$ , where  $\gamma$  is the Euler–Mascheroni constant.*

*Proof.* Since  $\varphi(n) \leq n$  for each positive integer  $n$ , it results  $m \geq \lfloor x \rfloor$ . In particular,  $m \rightarrow +\infty$  as  $x \rightarrow +\infty$ . Therefore, since the minimal order of  $\varphi(n)$  is  $e^{-\gamma} n / \log \log n$  (see [Ten95, Chapter I.5, Theorem 4]), we obtain

$$(e^{-\gamma} + o(1)) \frac{m}{\log \log m} \leq \varphi(m) \leq x,$$

as  $x \rightarrow +\infty$ . Now  $\varphi(n) \geq \sqrt{n}$  for each integer  $n \geq 7$ , thus  $m \leq x^2$  for  $x \geq 7$ . Hence,

$$m \leq (e^\gamma + o(1)) x \log \log m \leq (e^\gamma + o(1)) x \log \log(x^2) = (e^\gamma + o(1)) x \log \log x,$$

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as  $x \rightarrow +\infty$ .

On the other hand, let  $p_1 < p_2 < \dots$  be the sequence of all the (natural) prime numbers and let  $a_1 < a_2 < \dots$  be the sequence of all the 3-smooth numbers, i.e., the natural numbers of the form  $2^a 3^b$ , for some integers  $a, b \geq 0$ . Moreover, let  $s = s(x)$  be the greatest positive integer such that

$$(p_1 - 1) \cdots (p_s - 1) \leq \sqrt{x},$$

and let  $t = t(x)$  be the greatest positive integer such that

$$a_t(p_1 - 1) \cdots (p_s - 1) \leq x.$$

Note that  $s, t \rightarrow +\infty$  as  $x \rightarrow +\infty$ . Now we have (see [Ten95, Chapter I.1, Theorem 4])

$$\sqrt{x} < (p_1 - 1) \cdots (p_{s+1} - 1) < p_1 \cdots p_{s+1} \leq 4^{p_{s+1}},$$

hence

$$(2) \quad p_s > \frac{1}{2} p_{s+1} > \frac{1}{4 \log 4} \log x,$$

from Bertrand's postulate. Put  $m' := a_t p_1 \cdots p_s$ , so that for  $s \geq 2$  we get

$$\varphi(m') = a_t(p_1 - 1) \cdots (p_s - 1) \leq x,$$

and thus  $m \geq m'$ . By a result of Pólya [Pól18],  $a_t/a_{t+1} \rightarrow 1$  as  $t \rightarrow +\infty$ . Therefore, from our hypothesis on  $s$  and  $t$ , Mertens' formula [Ten95, Chapter I.1, Theorem 11] and (2) it follows that

$$\begin{aligned} m \geq m' &= \frac{a_t}{a_{t+1}} \cdot a_{t+1} \prod_{i=1}^s (p_i - 1) \cdot \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right)^{-1} > (1 + o(1)) \cdot x \cdot \frac{\log p_s}{e^{-\gamma} + o(1)} \\ &> (e^\gamma + o(1)) x \log \log x, \end{aligned}$$

as  $x \rightarrow +\infty$ . □

Actually, we do not make use of Lemma 2.1. We need more control on the factorization of a “large” positive integer  $m$  such that  $\varphi(m) \leq x$ , even at the cost of having only a lower bound for  $m$  and not an asymptotic formula.

**Lemma 2.2.** *For each  $x \geq 1$  there exists a positive integer  $m = m(x)$  such that:  $\varphi(m) \leq x$ ;  $m = 2^t Q$ , where  $t$  is a nonnegative integer and  $Q$  is an odd squarefree number; and*

$$m \geq (\tfrac{1}{2}e^\gamma + o(1)) x \log \log x,$$

as  $x \rightarrow +\infty$ .

*Proof.* The proof proceeds as the second part of the proof of Lemma 2.1, but with  $a_k := 2^{k-1}$  for each positive integer  $k$ . So instead of  $a_t/a_{t+1} \rightarrow 1$ , as  $t \rightarrow +\infty$ , we have  $a_t/a_{t+1} = 1/2$  for each  $t$ . We leave the remaining details to the reader. □

To study  $\Lambda_n$  is useful to consider the positive integers as a poset ordered by the divisibility relation  $|$ . Thus, obviously,  $\Lambda_n$  is a monotone nondecreasing function, i.e.,  $\Lambda_m | \Lambda_n$  for each positive integers  $m \leq n$ . The next lemma says that  $\Lambda_n$  is also super-multiplicative.

**Lemma 2.3.** *We have  $\Lambda_m \Lambda_n | \Lambda_{mn}$ , for any positive integers  $m$  and  $n$ .*

*Proof.* It is an easy exercise to prove that

$$\Lambda_n = \prod_{p \leq n} p^{\lfloor \log_p n \rfloor},$$

for each positive integer  $n$ , where  $p$  runs over all the prime numbers not exceeding  $n$ . Therefore, the claim follows since

$$\lfloor \log_p m \rfloor + \lfloor \log_p n \rfloor \leq \lfloor \log_p m + \log_p n \rfloor = \lfloor \log_p mn \rfloor,$$

for each prime number  $p$ . □

We recall some basic facts about cyclotomic polynomials. For each positive integer  $n$ , the  $n$ -th cyclotomic polynomial  $\Phi_n(x)$  is defined by

$$\Phi_n(x) := \prod_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} (x - e^{2\pi i k/n}).$$

It results that  $\Phi_n(x)$  is a polynomial with integer coefficients and that it is irreducible over the rationals, with degree  $\varphi(n)$ . We have the polynomial identity

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

where  $d$  runs over all the positive divisors of  $n$ . Moreover, it holds  $\Phi_n(a) \leq (a+1)^{\varphi(n)}$ , for all  $a \geq 0$ . The next lemma regards when  $\Phi_m(a)$  and  $\Phi_n(a)$  are not coprime.

**Lemma 2.4.** *Suppose that  $\gcd(\Phi_m(a), \Phi_n(a)) > 1$  for some integers  $m, n, a \geq 1$ . Then  $m/n$  is a prime power, i.e.,  $m/n = p^k$  for a prime number  $p$  and an integer  $k$ .*

*Proof.* See [Ge08, Theorem 7]. □

Finally, we state an useful lower bound for the sum of digits of the multiples of  $b^m - 1$ .

**Lemma 2.5.** *For each positive integers  $m$  and  $q$  it results  $s_b((b^m - 1)q) \geq m$ .*

*Proof.* See [BD12, Lemma 1]. □

### 3. PROOF OF THEOREM 1.1

Without loss of generality, we can assume  $n$  sufficiently large. Put  $x := \frac{1}{8} \log_{b+1} n \geq 1$ . Thanks to Lemma 2.2, we know that there exists a positive integer  $m$  such that  $\varphi(m) \leq x$  and

$$(3) \quad m > \frac{1}{3} e^\gamma x \log \log x > C_b \log n \log \log \log n,$$

where  $C_b > 0$  is a constant depending only on  $b$ . Precisely, we can assume that  $m = 2^t Q$ , where  $t$  is a nonnegative integer and  $Q$  is an odd squarefree number. Fix a nonnegative integer  $j \leq t$ . For each positive divisor  $d$  of  $Q$ , we have  $\varphi(2^{t-j}d) \mid \varphi(m/2^j)$  and so, a fortiori,  $\varphi(2^{t-j}d) \leq \varphi(m/2^j)$ . Therefore,

$$(4) \quad \Phi_{2^{t-j}d}(b) \leq (b+1)^{\varphi(2^{t-j}d)} \leq (b+1)^{\varphi(m/2^j)} \leq (b+1)^{\varphi(m)/2^{j-1}} \leq n^{1/2^{j+2}}.$$

Let  $\mu$  be the Möbius function. Now from (4) and Lemma 2.4 we have that the  $\Phi_{2^{t-j}d}(b)$ 's, where  $d$  runs over the positive divisors of  $Q$  such that  $\mu(d) = 1$ , are pairwise coprime and not exceeding  $n^{1/2^{j+2}}$ , thus

$$(5) \quad \prod_{\substack{d|Q \\ \mu(d)=1}} \Phi_{2^{t-j}d}(b) = \text{lcm}\{\Phi_{2^{t-j}d}(b) : d|Q, \mu(d)=1\} \mid \Lambda_{\lfloor n^{1/2^{j+2}} \rfloor}.$$

Similarly, the same result holds for the divisors  $d$  such that  $\mu(d) = -1$ . Clearly, we have

$$b^m - 1 = \prod_{d|m} \Phi_d(b) = \prod_{\substack{0 \leq j \leq t \\ r \in \{-1, +1\}}} \prod_{\substack{d|Q \\ \mu(d)=r}} \Phi_{2^{t-j}d}(b).$$

Moreover,

$$\left( \prod_{0 \leq j \leq t} \lfloor n^{1/2^{j+2}} \rfloor \right)^2 \leq \prod_{0 \leq j \leq t} n^{1/2^{j+1}} \leq n.$$

As a consequence, from (5) and Lemma 2.3, we obtain

$$b^m - 1 \mid \left( \prod_{0 \leq j \leq t} \Lambda_{\lfloor n^{1/2^{j+2}} \rfloor} \right)^2 \mid \Lambda_n.$$

Thus  $b^m - 1 \mid \Lambda_n$  and also  $b^m - 1 \mid n!$ , since obviously  $\Lambda_n \mid n!$ . In conclusion, from Lemma 2.5 and (3), we get

$$s_b(\Lambda_n), s_b(n!) \geq m > C_b \log n \log \log \log n,$$

which is our claim, this completes the proof.

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